Lecture 17 - Gyroscopes

A Puzzle...

We have seen throughout class that the center of mass is a very powerful tool for evaluating systems. However, don't let yourself get carried away with how useful it can be!

Give a counter-example to the following ***incorrect*** statement:
The gravitational force from an extended body of mass M equals the gravitational force from a point mass M located at the center of mass.

Solution
Truthfully, you would be hard pressed to find an instance where the above statement *is* correct. As a simple counter-example in 2D, consider the case of a mass m at $(1, 0)$ and another mass m at $(-1, 0)$. If we put another mass exactly between them, then the gravitational force on this mass would be zero (the contributions from both masses would exactly cancel). However, the gravitational force from a mass $2\,m$ at their center would certainly not be zero at this point (it would be infinite!)

Another very famous counter-example is the spherical shell of mass M, radius R, and uniform mass density. A well known fact (and one that we prove below in the section "Gravity in a Spherical Shell" below) is that the gravitational force from this spherical shell inside the shell is exactly zero at all points! This remarkable fact leads to an easy counter-example, since the actual gravitational force inside the shell is zero, while the gravitational force from a point mass m located at its center would not be zero.

Gyroscopes

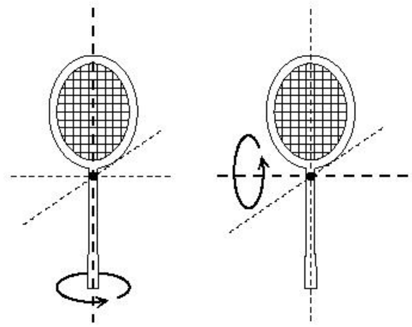
Gyroscopes...are really cool!

As was hinted earlier in the course, gyroscopes are one of the awesome phenomena (along with rolling cones, bicycles, etc.) that occur when we let the direction of angular momentum change in

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{1}$$

These ideas will be delved into much more deeply in junior level Classical Mechanics. Let me give one more example to whet your appetite.

Try spinning a tennis racket (or a book, etc.) about the axis of its handle (shown on the left in the figure below). If you start it off straight, it will rotate very nicely. However, if you try rotating the tennis racket as shown on the right in the figure below, you will never be able to get it spinning nicely. This awesome idea known as the ***Tennis Racket Theorem***. (Here are links to a slow motion video and to this effect seen in zero gravity.)



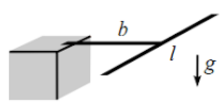
Problems

Projectile Motion

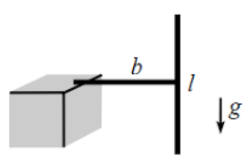
A ball is thrown at speed v from zero height on level ground. At what angle should it be thrown so that the area under the trajectory is maximum?

Falling Stick

A massless stick of length b has one end attached to a pivot and the other end glued perpendicularly to the middle of a stick of mass m and length l.
1. If the two sticks are held in a horizontal plane and then released, what is the initial acceleration of the center of mass?



2. If the two sticks are held in a vertical plane and then released, what is the initial acceleration of the center of mass?



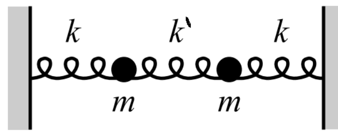
Escape Velocity

1. A rocket is shot radially outwards away from Earth. What would its initial velocity need to be for it to escape out to $r = \infty$?
2. What is the escape velocity if the rocket is shot tangentially from the surface of the Earth?

Normal Modes

Two masses m are each connected to a wall by a spring with spring constant k. The two masses are connected to each other by a spring with spring constant k'. Initially, all springs are unstretched and the masses are at rest.

While the general motion of this system can be very complicated, there are two very simple motions that we can easily analyze.
1. If both masses are shifted to the right by a distance A and then released from rest, what is the frequency of the resulting oscillation?
2. If the left mass is shifted *right* by a distance A and the right mass is shifted *left* by a distance A, what is the frequency of the resulting oscillation?



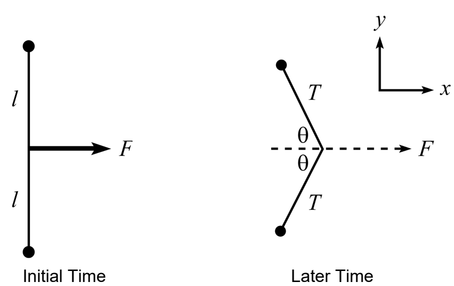
A Different Potential

A particle of mass m moves in a radial potential given by $V[r] = \beta\, r^k$ with angular momentum L. Find the radius of circular orbit in terms of L, m, k, and β.

Advanced Section: Crazy Chain

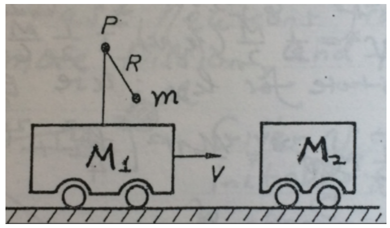
Advanced Section: Inelastic Collisions from Multiple Perspectives

A massless string of length $2\,l$ connects two hockey pucks that lie on frictionless ice. A constant horizontal force F is applied to the midpoint of the string, perpendicular to it. How much kinetic energy is lost when the pucks collide, assuming they stick together?



Advanced Section: Train Wreck

<u>Example</u>
A cart of mass M_1 has a pole on it from which a ball of mass m hangs from a thin string of negligible mass and length R attached at a point P. The cart and ball have initial velocity V (the ball is initially at rest with respect to the cart, so it hangs straight down). The cart crashes into another cart of mass M_2 and sticks to it. Assume that $m \ll M_1, M_2$.
1. Find the velocity V' of the two carts after the collision
2. Find the smallest initial velocity V so that the ball will complete a circle around the point P after the collision



Advanced Section: A Geometric Series of Bounces

Advanced Section: Turning Rope

Advanced Section: Oscillations and Rotations

Gravity in a Spherical Shell

Prove that the gravitational force inside of a spherical shell is zero.

Solutions

Projectile Motion

If the projectile is launched at angle θ, then

$$x = (v \operatorname{Cos}[\theta])\, t \tag{22}$$

$$y = (v \operatorname{Sin}[\theta])\, t - \frac{1}{2} g\, t^2 \tag{23}$$

The projectile starts off at $x_{\text{min}} = 0$ at $t = 0$. The maximum distance (when $y = 0$) occurs at $t_{\text{max}} = \frac{2 v \operatorname{Sin}[\theta]}{g}$ at which point the projectile has traveled a distance $x_{\text{max}} = (v \operatorname{Cos}[\theta])\, t_{\text{max}} = \frac{v^2 \operatorname{Sin}[2\theta]}{g}$. We can solve the above equation for $t = \frac{x}{v \operatorname{Cos}[\theta]}$ which we can then substitute into the y equation,

$$\begin{aligned} y &= (v \operatorname{Sin}[\theta]) \left(\frac{x}{v \operatorname{Cos}[\theta]}\right) - \frac{1}{2} g \left(\frac{x}{v \operatorname{Cos}[\theta]}\right)^2 \\ &= \operatorname{Tan}[\theta]\, x - \frac{g}{2 v^2 \operatorname{Cos}[\theta]^2} x^2 \end{aligned} \tag{24}$$

This allows us to integrate and compute the total area (A) under the curve of the projectile motion

$$\begin{aligned} A &= \int_{x_{\text{min}}}^{x_{\text{max}}} \left(\operatorname{Tan}[\theta]\, x - \frac{g}{2 v^2 \operatorname{Cos}[\theta]^2} x^2\right) dx \\ &= \left(\frac{\operatorname{Tan}[\theta]}{2} x^2 - \frac{g}{6 v^2 \operatorname{Cos}[\theta]^2} x^3\right)_{x = x_{\text{min}}}^{x = x_{\text{max}}} \\ &= \frac{\operatorname{Tan}[\theta]}{2} \left(\frac{v^2 \operatorname{Sin}[2\theta]}{g}\right)^2 - \frac{g}{6 v^2 \operatorname{Cos}[\theta]^2} \left(\frac{v^2 \operatorname{Sin}[2\theta]}{g}\right)^3 \\ &= \frac{2 v^4}{3 g^2} \operatorname{Sin}[\theta]^3 \operatorname{Cos}[\theta] \end{aligned} \tag{25}$$

We can now integrate the area with respect to θ and set it equal to 0 to maximize it,

$$\begin{aligned} 0 &= \frac{dA}{d\theta} \\ &= \frac{2 v^4}{3 g^2} \left(3 \operatorname{Sin}[\theta]^2 \operatorname{Cos}[\theta]^2 - \operatorname{Sin}[\theta]^4\right) \end{aligned} \tag{26}$$

which occurs when $\operatorname{Tan}[\theta] = \sqrt{3}$ or equivalently $\theta = \frac{\pi}{3}$, which is the desired answer.

Note that the maximal area equals $A_{\max} = \frac{\sqrt{3}}{8} \frac{v^4}{g^2}$. By dimensional analysis, the area must have been proportional to $\frac{v^4}{g^2}$! □

Falling Stick

1. Calculate τ and L about the pivot point. The torque is due to gravity, which acts on the center of mass with magnitude $m\,g\,b$. Using the parallel axis theorem, the moment of inertia around the horizontal axis through the pivot (and perpendicular to the massless stick) equals $m\,b^2$. When the stick starts to fall, $\tau = \frac{dL}{dt} = I\,\alpha$ yields

$$m\,g\,b = (m\,b^2)\,\alpha \tag{27}$$

Therefore, the initial acceleration of the center of mass equals $b\,\alpha = g$. This makes sense because the stick initially falls straight down, and the pivot provides no force because it does not yet know that the stick is falling.

2. The only difference now is that the moment of inertia equals $\frac{1}{12}\,m\,l^2 + m\,b^2$. Therefore, $\tau = \frac{dL}{dt} = I\,\alpha$ yields

$$m\,g\,b = \left(\frac{1}{12}\,m\,l^2 + m\,b^2\right)\alpha \tag{28}$$

so that the initial acceleration of the center of mass equals

$$b\,\alpha = \frac{g}{1+\frac{l^2}{12\,b^2}} \tag{29}$$

For $l \ll b$, this goes to $b\,\alpha \to g$, which makes sense (i.e. just like for a point mass). For $l \gg b$, it goes to zero, which makes sense because a tiny movement of the center of mass corresponds to a very large movement of the ends of the large stick. Thus, by conservation of energy, the center of mass must move very slowly.

Escape Velocity

<u>Solution</u>

1. The energy of the rocket on the Earth's surface would be

$$E = \frac{1}{2}\,m\,v_{\text{escape}}^2 - \frac{G\,M_E\,m}{R_E} \tag{30}$$

The minimum energy required for this rocket to escape out to infinity would require for it to just barely reach $r = \infty$ with no remaining kinetic energy (i.e. it must turn all of its kinetic energy into translational energy to reach $r = \infty$). Therefore we must have

$$E = 0 - \frac{G\,M_E\,m}{\infty} = 0 \tag{31}$$

So the defining condition for escape velocity is $E = 0$, which gives us a simple way to solve for escape velocity from Earth's surface

$$0 = \frac{1}{2}\,m\,v_{\text{escape}}^2 - \frac{G\,M_E\,m}{R_E} \tag{32}$$

Using $M_E = 6 \times 10^{24}$ kg and $R_E = 6.4 \times 10^6\,m$, we find $v_{\text{escape}} = \left(\frac{2\,G\,M_E}{R_E}\right)^{1/2} = 11\,100\,\frac{m}{s}$.

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M = 6 × 10^24; R = 6.4 × 10^6; G = 6.67 * 10^-11; m = 1;
Sqrt[2 G M / R]
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2. More generally, if the rocket was shot in any direction relative to the Earth's surface, its orbit motion would follow the orbit equation we derived last time,

$$r = \frac{L^2}{G\,M\,m^2}\,\frac{1}{1+\epsilon\,\text{Cos}[\theta]} \tag{33}$$

$$\epsilon = \left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{34}$$

An orbit with $\epsilon \geq 1$ would escape out to infinity, which requires $E \geq 0$. Thus, the minimum energy needed to escape is $E = 0$, which implies the same escape velocity $v_{\text{escape}} = \left(\frac{2\,G\,M_E}{R_E}\right)^{1/2}$ found above. In other words, having a speed v_{escape} pointing in any direction at the surface of the Earth allows you to escape to $r = \infty$ (provided that your orbit does not intersect the Earth, in which case you will crash and burn). □

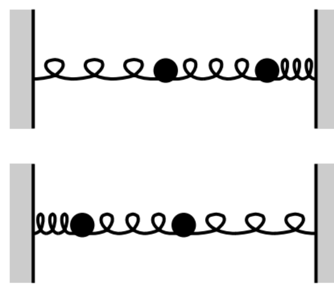
Normal Modes

1. If we ignore the spring in the middle for now, the left and right masses would oscillate about their equilibrium positions (defining the right as positive) as

$$x[t] = A\,\text{Cos}[\omega_0\,t] \tag{35}$$

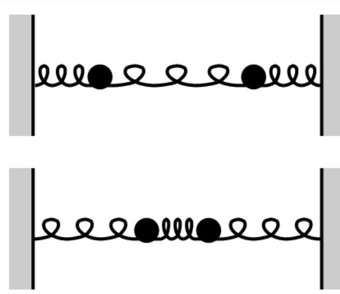
where

$$\omega_0 = \left(\frac{k}{m}\right)^{1/2} \tag{36}$$



How does the middle spring effect this motion? The distances between the two masses remains fixed at all times, so the middle spring remains unstretched and hence does not do anything. Therefore, the above expression described the displacement of both masses, and each mass oscillates with the frequency $\left(\frac{k}{m}\right)^{1/2}$.

2. By the symmetry in this problem, the two masses will oscillate inwards and outwards at the same time.



How can we calculate the frequency of oscillation? Consider the displacement $x_l[t]$ of the left mass (since by symmetry $x_r[t] = -x_l[t]$). The force on the left mass will be

$$m\,\ddot{x}_l = -k\,x_l - k'\,(2\,x_l) \tag{37}$$

where the first term $-k\,x_l$ is the normal spring force from the left spring and the term $-k'\,(2\,x_l)$ is the spring force from the middle spring; the factor of 2 comes from the fact that when the left mass moves by a distance x_l and compressed the middle spring by that amount, the right mass also moves inward by an amount x_l and compressed the middle spring by the same amount. Simplifying the above expression,

$$m\,\ddot{x}_l = -(k + 2\,k')\,x_l \tag{38}$$

which has the solution (with initial conditions substituted in)

$$x_l[t] = A\,\mathrm{Cos}[\omega\, t] \tag{39}$$

where

$$\omega = \left(\frac{k+2\,k'}{m}\right)^{1/2} \tag{40}$$

In this case, the masses are oscillating against each other at a larger frequency than in Case 1. □

As you will learn in more advanced courses, any motion of the two masses (regardless of how complicated it looks) can be broken down into a superposition of these two motions above. These two types of motions are called the *normal modes* of the system.

A Different Potential

The outwards radial force is given by

$$\vec{F} = -\frac{dV}{dr}\,\hat{r} = -k\,\beta\,r^{k-1}\,\hat{r} \tag{41}$$

For circular motion, the inwards radial force must equal $\frac{m\,v^2}{r}$,

$$k\,\beta\,r^{k-1} = \frac{m\,v^2}{r} \tag{42}$$

For circular motion, $L = m\,r\,v$ so that $\frac{m\,v^2}{r} = \frac{L^2}{m\,r^3}$. Substituting this in and solving for r,

$$k\,\beta\,r^{k-1} = \frac{L^2}{m\,r^3} \tag{43}$$

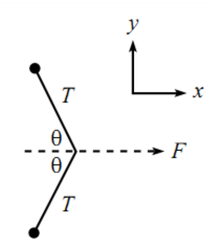
$$r = \left(\frac{L^2}{m\,k\,\beta}\right)^{1/(k+2)} \tag{44}$$

Note that if k is negative, then β must also be negative (as in the case of the gravitational potential) for there to be a real solution for this circular orbit. □

Advanced Section: Crazy Chain

Advanced Section: Inelastic Collisions from Multiple Perspectives

We will solve this in three ways, each way more slick then the last.



Solution 1

Since the string is massless, the forces on the midpoint of the string must sum to zero. Balancing forces in the x-direction,

$$F = 2\,T\,\mathrm{Cos}[\theta] \tag{51}$$

The bottom mass feels a the tension force pulling it upwards and to the right. The upwards component of this force equals

$$m\,a_y = T\,\mathrm{Sin}[\theta] = \frac{F}{2}\,\mathrm{Tan}[\theta] \tag{52}$$

When the bottom mass moves a distance y upwards, it is now $l - y$ away from its collision height with the other mass. Because the length from the bottom mass to the point of application of the force is l, this implies that

$\text{Sin}[\theta] = \frac{l-y}{l}$. Simple geometry then tells us that

$$\text{Tan}[\theta] = \frac{l-y}{\sqrt{l^2-(l-y)^2}} \tag{53}$$

We use the trick $a_y = \frac{dv_y}{dt} = \frac{dv_y}{dy}\frac{dy}{dt} = v_y \frac{dv_y}{dy}$ and substitute into the above equation to find

$$m\, v_y \frac{dv_y}{dy} = m\, a_y = \frac{F}{2}\text{Tan}[\theta] = \frac{F}{2}\frac{l-y}{\sqrt{l^2-(l-y)^2}} \tag{54}$$

Rearranging,

$$m\, v_y\, dv_y = \frac{F}{2}\frac{l-y}{\sqrt{l^2-(l-y)^2}}\, dy \tag{55}$$

Integrating,

$$\int_0^{v_f} m\, v_y\, dv_y = \frac{F}{2}\int_0^l \frac{l-y}{\sqrt{l^2-(l-y)^2}}\, dy \tag{56}$$

$$\frac{1}{2} m\, v_f^2 = \frac{F}{2}\left(\sqrt{y\,(2\,l-y)}\right)_{y=0}^{y=l} = \frac{F\,l}{2} \tag{57}$$

This entire kinetic energy is lost by the bottom puck when it collides with the top puck (the velocity in the x-direction is not effected). Since the top puck loses the same amount of kinetic energy, the total energy lost equals $F\,l$.

Solution 2

Solution 1 was straightforward, although the integral with y was pretty nasty. As a way to avoid that complicated-looking function, we can instead proceed as follows

$$m\, v_y \frac{dv_y}{dy} = \frac{F}{2}\text{Tan}[\theta] \tag{58}$$

which can be rearranged to obtain

$$m\, v_y\, dv_y = \frac{F}{2}\text{Tan}[\theta]\, dy \tag{59}$$

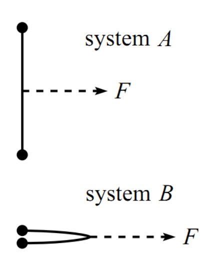
Integrating and changing variables from y to θ using $y = l - l\,\text{Sin}[\theta]$

$$\begin{aligned}\int_0^{v_f} m\, v_y\, dv_y &= \frac{F}{2}\int_0^l \text{Tan}[\theta]\, dy \\ &= \frac{F}{2}\int_{\pi/2}^0 \text{Tan}[\theta]\, d(l - l\,\text{Sin}[\theta]) \\ &= -\frac{F\,l}{2}\int_{\pi/2}^0 \text{Sin}[\theta]\, d\theta \\ &= \frac{F\,l}{2}(\text{Cos}[\theta])_{\theta=\pi/2}^{\theta=0} \\ &= \frac{F\,l}{2}\end{aligned} \tag{60}$$

which recoups the answer found above.

Solution 3 (Super slick)

The incredible simplicity of the solution demands an equally simple explanation. Consider two systems, A and B where A is the original setup, while B starts with θ already at zero.



Let the pucks in both systems start simultaneously at $x = 0$. As the force F is applied, all four pucks will have the same $x[t]$, because the same force in the x-direction, namely $F/2$, is being applied to every puck at all times. After the collision, both systems will therefore look exactly the same. Let the collision of the pucks occur at $x = d$. At this point, $F\,(d + l)$ work has been done on system A, because the center of the string (where the force is applied) ends up moving a distance l more than the masses. However, only $F\,d$ work has been done on system B. Since both systems have the same kinetic energy after the collision, the extra $F\,l$ work done on system A must be what is lost in the collision.

Note that this reasoning makes it clear that this $F\,l$ result holds even if we have many masses distributed along the string, or if we have a rope with a continuous mass distribution. The only requirement is that the mass be symmetrically distributed around the midpoint. □

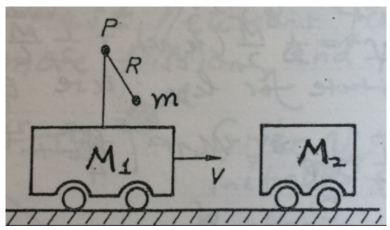
Advanced Section: Train Wreck

1. Using conservation of linear momentum, the final velocity V' of the two carts stuck together satisfies

$$M_1\,V = (M_1 + M_2)\,V' \tag{61}$$

$$V' = \frac{M_1}{M_1+M_2}\,V \tag{62}$$

where we have ignored the tiny mass m because it will continue to move at velocity V after the collision.



2. After the collision, the mass m undergoing both translational and circular motion. Thus, we can greatly simplify this problem by working in the frame moving at velocity V' after the collision (i.e. the rest frame of the two carts after the collision), where the mass m is purely undergoing circular motion about P.

As stated above, the velocity of the mass m will be unchanged immediately before and after the collision, since there are no sideways forces acting on this mass during the collision. Hence its velocity at the bottom of the circle v_{bottom} immediately after the collision is given by

$$v_{\text{bottom}} = V - V' = \frac{M_2}{M_1+M_2}\,V \tag{63}$$

How do we determine the minimal speed necessary for the mass m to undergo circular motion? At the very top of the circle, both gravity ($m\,g$) and the tension force (T) point downwards, and together these two forces must equal to the net inwards force $\frac{m\,v_{\text{top}}^2}{R}$ where v_{top} is the speed of the mass. The minimum possible value of the tension is $T = 0$ (since the tension cannot be negative for a string, which would imply that the string is pushing the ball up). Therefore, the minimum velocity at the top of the circle must satisfy

$$m\,g = \frac{m\,v_{\text{top}}^2}{R} \tag{64}$$

$$v_{\text{top}} = (g\,R)^{1/2} \tag{65}$$

We can related v_{bottom} and v_{top} using the conservation of energy,

$$\frac{1}{2}\,m\,v_{\text{bottom}}^2 = \frac{1}{2}\,m\,v_{\text{top}}^2 + m\,g\,(2\,R) \tag{66}$$

$$v_{\text{bottom}}^2 = v_{\text{top}}^2 + 4\,g\,R \tag{67}$$

$$v_{\text{bottom}} = (5\,g\,R)^{1/2} \tag{68}$$

where in the last step we substitute in Equation (65).

As a quick recap of what we have done: Equation (68) represents the minimum possible speed that m can have at the bottom point of its arc so that it will undergo circular motion. On the other hand, Equation (63) represents the actual speed of m at the bottom of its arc right after the collision due to conservation of energy. Therefore, by setting these two expressions equal to each other, we find the minimum speed V of the initial cart so that m will undergo circular motion, namely

$$\frac{M_2}{M_1+M_2} V = (5\, g\, R)^{1/2} \tag{69}$$

$$V = \left(1 + \frac{M_1}{M_2}\right) (5\, g\, R)^{1/2} \tag{70}$$

We can quickly check that the dimensions of this result make sense. We also see that the larger the ratio $\frac{M_1}{M_2}$ is, the faster the initial speed has to be. This makes sense, because the larger M_1 is, the less it will be slowed down by M_2, resulting in a smaller initial velocity v_{bottom} for mass m. □

Advanced Section: A Geometric Series of Bounces

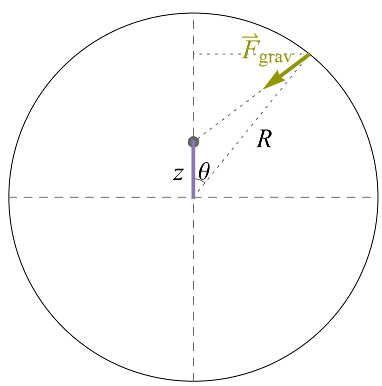
Advanced Section: Turning Rope

Advanced Section: Oscillations and Rotations

Gravity in a Spherical Shell

Solution 1

The most natural approach is straight-up integration. Consider a point mass m that (without loss of generality) lies on the z-axis at coordinate (0, 0, z) where $0 \le z < R$. By radial symmetry, the gravitational force must point in the z-direction, so we will only focus on the component of gravitational force in this direction.



Assuming a uniform mass density σ for the sphere, the gravitational force equals

$$\vec{F}_{\text{grav}}[z] = \hat{z} \int_0^{2\pi} \int_0^{\pi} -\frac{G\, m \left(\sigma\, R^2\, \text{Sin}[\theta]\, d\theta\, d\phi\right)}{R^2+z^2-2\, R\, z\, \text{Cos}[\theta]} \frac{z-R\, \text{Cos}[\theta]}{\left(R^2+z^2-2\, R\, z\, \text{Cos}[\theta]\right)^{1/2}} \tag{110}$$

where the first term is the typical $-\frac{G\, m\, M}{r^2}$ gravitational force and the second term pulls out the z-component of this force. The ϕ integral is straightforward and just yields a factor of $2\,\pi$. The θ integral, while not trivial, has a simple answer.

$$\begin{aligned}\vec{F}_{\text{grav}}[z] &= \hat{z}\left(-G\,m\,\sigma\,R^2\right)\int_0^{2\pi}\int_0^{\pi}\frac{\text{Sin}[\theta]\,(z-R\,\text{Cos}[\theta])}{\left(R^2+z^2-2\,R\,z\,\text{Cos}[\theta]\right)^{3/2}}\,d\theta\,d\phi\\
&= \hat{z}\left(-2\,\pi\,G\,m\,\sigma\,R^2\right)\int_0^{\pi}\frac{\text{Sin}[\theta]\,(z-R\,\text{Cos}[\theta])}{\left(R^2+z^2-2\,R\,z\,\text{Cos}[\theta]\right)^{3/2}}\,d\theta\\
&= \hat{z}\left(-2\,\pi\,G\,m\,\sigma\,R^2\right)\left(\frac{R-z\,\text{Cos}[\theta]}{z^2\left(R^2+z^2-2\,R\,z\,\text{Cos}[\theta]\right)^{1/2}}\right)\Bigg|_{\theta=0}^{\theta=\pi}\\
&= \hat{z}\left(-2\,\pi\,G\,m\,\sigma\,R^2\right)\left(\frac{R+z}{z^2(R+z)}-\frac{R-z}{z^2\,|R-z|}\right)\\
&= 0\end{aligned}$$

That's a killer result! Of course, we should double check our integration using *Mathematica.*

$$\texttt{Integrate}\left[-\frac{\texttt{G m}\left(\sigma\,\texttt{R}^2\,\texttt{Sin}[\theta]\right)\,(\texttt{z - R Cos}[\theta])}{\left(\texttt{R}^2+\texttt{z}^2-\texttt{2 R z Cos}[\theta]\right)^{3/2}},\ \{\theta,0,\pi\},\ \{\phi,0,2\pi\},\ \texttt{Assumptions}\to 0<\texttt{z}<\texttt{R}\right]$$

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0
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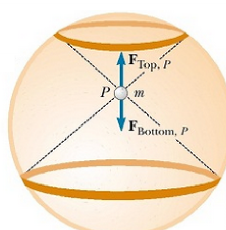
What this means is that inside of a spherical shell, you do not feel any gravitational force! It is as if the shell does not exist at all. (The situation is quite different if you are on are outside of the shell, however.) The next solution explores some of the symmetries of the sphere that enable this miraculous result.

Solution 2

We can show that inside the shell the gravitational force equals zero by showing that it cancels along each pair of circular rings lying between θ and $\theta + d\theta$. Orient the origin at the point that we wish to analyze, and denote the distance from the origin to the center of the sphere as a. Call the horizontal and vertical axes as the x-axis and z-axis. The equation of the circle is $x^2 + (z + a)^2 = R^2$



We will just consider the cut at $y = 0$. The equation of the great circle in Cartesian coordinates equals

$$x^2 + (z-a)^2 = R^2 \tag{112}$$

Thus, the equation of the circle in spherical coordinates is

$$r^2\,\text{Sin}[\theta]^2 + (r\,\text{Cos}[\theta] - a)^2 = R^2 \tag{113}$$

which we can simplify to

$$r^2 - (2\,a\,\text{Cos}[\theta])\,r + (a^2 - R^2) = 0 \tag{114}$$

which we can solve for $r[\theta]$,

$$r[\theta] = -a\,\text{Cos}[\theta] + \sqrt{R^2 - a^2\,\text{Sin}[\theta]^2} \tag{115}$$

Consider the upper band between angle θ and $\theta + d\theta$. The radius of the band equals $r[\theta]\,\text{Sin}[\theta]$, and the distance of each point on the band from the origin is $r[\theta]^2$. The width of the band is given by $\sqrt{(dx)^2 + (dz)^2}$ which we calculate using

$$x = r\,\text{Sin}[\theta] \tag{116}$$

$$z = r\,\text{Cos}[\theta] \tag{117}$$

Differentiating,

$$d x = d r \,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\, d\theta \tag{118}$$
$$d z = d r \,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\, d\theta \tag{119}$$

Thus the width equals

$$\sqrt{(d x)^2 + (d z)^2} = \sqrt{(d r)^2 + r^2 (d\theta)^2} = \sqrt{r'[\theta]^2 + r^2}\; d\theta \tag{120}$$

We can now calculate the gravitational force from the ring spanning the angle θ to $\theta + d\theta$. Note that the net gravitational force from this ring must (by symmetry) point in the z-direction. Assuming a uniform mass density σ, the total gravitational force from the ring from angle θ to $\theta + d\theta$ on a mass m equals

$$\begin{aligned} F_{\mathrm{grav}}[\theta] &= (G\, m\, \sigma)\, \frac{2\,\pi\,(r[\theta]\,\mathrm{Sin}[\theta])\,\sqrt{r[\theta]^2 + r'[\theta]^2}\; d\theta}{r[\theta]^2}\,\mathrm{Cos}[\theta] \\ &= G\, m\, \sigma\, \pi\, \mathrm{Sin}[2\,\theta]\left(1 + \left\{\frac{r'[\theta]}{r[\theta]}\right\}^2\right)^{1/2} d\theta \end{aligned} \tag{121}$$

where the factor of $\mathrm{Cos}[\theta]$ on the right comes from the portion of the gravitational force that points along the z-direction (all other components of the gravitational force will cancel around the ring by symmetry).

All that remains to show is that when you let $\theta \to \pi - \theta$ in the above equation, then $F_{\mathrm{grav}}[\theta] = -F_{\mathrm{grav}}\left[\frac{\pi}{2} - \theta\right]$, which would imply that the gravitational force from the ring between θ and $\theta + d\theta$ cancels the gravitational force from the ring between $-\theta$ and $-\theta - d\theta$. Indeed, we find that

$$\begin{aligned} F_{\mathrm{grav}}[\pi - \theta] &= G\, m\, \sigma\, \pi\, \mathrm{Sin}[2\,(\pi - \theta)]\left(1 + \left\{\frac{r'[\pi-\theta]}{r[\pi-\theta]}\right\}^2\right)^{1/2} d\theta \\ &= -G\, m\, \sigma\, \pi\, \mathrm{Sin}[2\,\theta]\left(1 + \left\{\frac{r'[\pi-\theta]}{r[\pi-\theta]}\right\}^2\right)^{1/2} d\theta \end{aligned} \tag{122}$$

In other words, we would like to show that

$$\left|\frac{r'[\theta]}{r[\theta]}\right| = \left|\frac{r'[\pi-\theta]}{r[\pi-\theta]}\right| \tag{123}$$

Carrying out the simple differentiation,

$$\frac{r'[\theta]}{r[\theta]} = \frac{a^2\,\mathrm{Sin}[\theta]^2}{R^2 - a^2\,\mathrm{Sin}[\theta]^2} \tag{124}$$

so that

$$\begin{aligned} \frac{r'[\pi-\theta]}{r[\pi-\theta]} &= \frac{a^2\,\mathrm{Sin}[\pi-\theta]^2}{R^2 - a^2\,\mathrm{Sin}[\pi-\theta]^2} \\ &= \frac{a^2\,\mathrm{Sin}[\theta]^2}{R^2 - a^2\,\mathrm{Sin}[\theta]^2} \\ &= \frac{r'[\theta]}{r[\theta]} \end{aligned} \tag{125}$$

This concludes the proof. This result is an astoundingly beautiful symmetry of the sphere. It requires both the $\frac{1}{r^2}$ gravitational force as well as the spherical shape. A truly magnificent result. However, this is not the only symmetry in a sphere, as the next solution shows.

Advanced Section: Another Symmetry of the Sphere

Mathematica Initialization